On Light Elliptically Polarised by Reflexion, especially near the Polarising Angle: a Comparison with Theory.[*]

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Taking the phenomena of light to be due to periodic displacements in a rotational ether, we shall investigate the exact character of these displacements in the case of reflection from an isotropic transparent substance. Two well-known vectors play the principal parts in any such optical discussion—the displacement of the medium and its curl; in the electro-magnetic theory the former is proportional to the magnetic force and the latter to the electric "displacement." To give precision to the discussion we shall deal throughout with the displacement of the medium, there being no difficulty in dealing in a similar fashion with the curl.

Let the displacement in the incident light be of unit amplitude and polarised in a plane making an angle θ with the plane of incidence. It can be resolved into two components $\xi = \cos\theta\cos pt$ in the plane of incidence, and $\eta = \sin\theta\cos pt$ perpendicular to this plane. The corresponding components in the reflected beam are then given by the formulæ

$$\xi = -\mathrm{R}'\cos\theta\cos(pt-\rho') \quad \text{and} \quad \eta = -\mathrm{R}\sin\theta\cos(pt-\rho).$$

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[*] [With reference to an oversight in his paper "On Metallic Reflexion," 'Roy. Soc. Proc.,' vol. 77, Jan., 1906, which has recently been pointed out privately by Mr. C. E. Hassé, Prof. Maclaurin writes that he had himself observed it immediately on receipt of the printed paper, but that fortunately the blemish does not invalidate any of the results, beyond requiring a slight modification of the argument in one place.

At the foot of p. 218, in quoting the formula $\mathrm{E} = \mu^{-2}\int_0^1 \mu\,dx$ from the previous paper, the first factor μ^{-2} was left out. When it is inserted, the modulus of E is no longer large, and the following modification becomes necessary. We have

$$r_1 = \frac{\mathrm{A}_1+a_1}{\mathrm{A}'_1+a_1'} = \frac{\mathrm{A}_1}{\mathrm{A}_1'}\left[1+\frac{2cd_1\cos\phi}{\mu^2\cos^2\phi-\cos^2\phi'}\{(\mathrm{E}-1)\mu^2+(\mathrm{F}-\mathrm{E})\sin^2\phi\}\right];$$

$$r_2 = \frac{\mathrm{A}_2}{\mathrm{A}_2'}\left[1+\frac{2cd_1\cos\phi}{\mu^2\cos^2\phi'-\cos^2\phi}(\mathrm{E}-1)\mu^2\right].$$

If both the terms $(\mathrm{E}-1)\mu^2$ and $(\mathrm{F}-\mathrm{E})\sin^2\phi$ are to be retained, we would require two complex constants to specify the layer of transition, which the experimental knowledge would hardly be adequate to determine. When, however, ϕ is small, the term $(\mathrm{F}-\mathrm{E})\sin^2\phi$ will be negligible in comparison with $(\mathrm{E}-1)\mu^2$; and even if ϕ is not small, the first term will not usually be more than one-tenth of the second, the modulus of the factor μ^2 in the latter being large. The neglect of the first term may therefore be expected to produce an error of about one-tenth in the estimated correction due to the layer of transition for large incidences, and practically no error at all for small incidences. This degree of approximation fortunately remains adequate, in relation to the exactness of the data utilised in the argument.]

Here R′ and R are given by the sine and tangent formulæ of Fresnel, viz.:—

$$R' = \frac{\sin(\phi-\phi')}{\sin(\phi+\phi')}; \quad R = \pm\frac{\tan(\phi-\phi')}{\tan(\phi+\phi')},$$

where ϕ and ϕ' are the angles of incidence and refraction, and the upper or lower sign is chosen so as to make R positive throughout. ρ and ρ' are the changes of phase produced by reflection. In the case of an abrupt transition from one medium to the other, ρ' would be zero throughout, while ρ would be zero or π according as ϕ were less or greater than Brewster's angle $\tan^{-1}\mu$. In the more usual circumstances of a gradual transition ρ and ρ' are given very approximately by the formulæ:—*

$$\tan\rho = d_1 \times \frac{2\mu^2\cos\phi\,[(F-E)\sin^2\phi-(1-E)\mu^2]}{(\mu^2-1)[\sin^2\phi-\mu^2\cos^2\phi]},$$

$$\tan\rho' = d_1 \times \frac{2\mu^2(1-E)\cos\phi}{\mu^2-1};$$

and the difference of phase $\Delta = \rho-\rho'$ by the formula

$$\tan\Delta = d_1 \times \frac{2\mu^2\{(F-1)-\mu^2(1-E)\}}{\mu^2-1}\cdot\frac{\sin\phi\tan\phi}{\tan^2\phi-\mu^2}$$

$$= -4\epsilon_0\frac{\mu^2}{(1+\mu^2)^{\frac{1}{2}}}\cdot\frac{\sin\phi\tan\phi}{\tan^2\phi-\mu^2}.$$

Here $d_1 = 2\pi d/\lambda$, where λ is the wave-length in air and d is the thickness of the transition layer; E and F are numbers depending on the law of distribution of the refractive index within the layer, and ϵ_0 is the coefficient of ellipticity at Brewster's angle.

The formula for $\tan\rho'$ shows that it is proportional to $\cos\phi$, so that it will diminish to zero as the incidence increases. In nearly all cases that have been examined E is less than unity and d_1 is a very small quantity. Thus ρ' will be small and positive throughout, approaching the limit zero as the incidence becomes grazing. At direct incidence, where there is no distinction between the plane of incidence and that at right angles to it, ρ is equal to ρ'.

With the substances that are dealt with specially in this paper, and in nearly all cases where the experimental determinations are well assured, ρ increases with the incidence, slowly everywhere except near Brewster's angle, in the neighbourhood of which its increase is very rapid. In such circumstances ρ is greater than ρ' and the crests of the waves representing ξ are reached sooner than the crests of the waves representing η. Thus η, the displacement perpendicular to the plane of incidence, lags behind ξ, the displacement parallel to that plane, the difference of phase being Δ.*

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* See 'Roy. Soc. Proc.,' A, vol. 76, 1905, pp. 55 and 57.

We have seen that the components of the displacement in the reflected beam are given by

$$\xi = -\mathrm{R}' \cos\theta \cos(pt-\rho') \quad \text{and} \quad \eta = -\mathrm{R}\sin\theta\cos(pt-\rho).$$

If $\theta = 0$ or $\frac{1}{2}\pi$, then $\eta = 0$ or $\xi = 0$, so that the reflected light is plane polarised in or perpendicular to the plane of incidence. In other cases, in general, it is elliptically polarised, the elliptic orbit being obtained by eliminating t from the above equations for ξ and η. In this way we obtain as the equation of the ellipse

$$\frac{\xi^2}{\mathrm{R}'^2\cos^2\theta} + \frac{\eta^2}{\mathrm{R}^2\sin^2\theta} - \frac{2\xi\eta\cos\Delta}{\mathrm{R}\mathrm{R}'\sin\theta\cos\theta} = \sin^2\Delta.$$

In the special cases of direct or grazing incidence the ellipse degenerates into a straight line. If $\Delta = 0$ the reflected light is plane polarised in an azimuth θ' given by $\tan\theta' = \dfrac{\eta}{\xi} = \dfrac{\mathrm{R}}{\mathrm{R}'}\tan\theta = \epsilon\tan\theta$, while if $\Delta = \pi$ the light is plane polarised in an azimuth θ', where $\tan\theta' = -\epsilon\tan\theta$. These two cases would represent the state of affairs below and above Brewster's angle respectively, if the transition from one medium to the other were abrupt, so that in such circumstances the reflected light would be plane polarised throughout.

By considering the signs of ξ, η, $\partial\xi/\partial t$ and $\partial\eta/\partial t$ in the formulæ above, it appears that the motion in the elliptic orbit takes place in a counter clockwise sense. To give precision to this statement we may express the matter thus:—Look along the ray (whether incident or reflected) in the direction of the propagation of the light. Turn counter clockwise from the plane of incidence through an acute angle θ until the direction of the incident vibrations is reached. The motion in the orbit is then clockwise.†

We proceed to consider the elements of the ellipse referred to above. Putting $\tan\alpha = \epsilon\tan\theta$, the equation of the ellipse becomes

$$\xi^2\tan^2\alpha + \eta^2 - 2\xi\eta\tan\alpha \,.\, \cos\Delta = \mathrm{R}^2\sin^2\Delta\sin^2\theta.$$

If ψ be the azimuth of the major axis, we have

$$\tan 2\psi = \frac{2h'}{a'-b'} = \frac{2\tan\alpha\cos\Delta}{1-\tan^2\alpha} = \tan 2\alpha \,.\, \cos\Delta.$$

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* Of course, if ρ' were greater than ρ, this statement would have to be reversed. Moreover, if we were dealing with the curl instead of the displacement, $(\mathrm{R}\rho)$ would be interchanged with $(\mathrm{R}'\rho')$, so that the component parallel to the plane of incidence would lag behind the other.

† If we were dealing with the curl, instead of the displacement, the motion would be counter clockwise.

The eccentricity (e) is given by

$$\frac{e^4}{(2-e^2)^2} = 1 - \frac{4\,(a'b'-h'^2)}{(a'+b')^2} = 1 - \sin^2 2\alpha \sin^2 \Delta.$$

Putting $\sin 2\gamma = \sin\Delta \sin 2\alpha$, this gives

$$e = \frac{\sqrt{(\cos 2\gamma)}}{\cos\gamma},$$

while for the semi-axes (a and b) we have

$$a = \frac{\mathrm{R}\sin\theta\cos\gamma}{\sin\alpha} \quad\text{and}\quad \frac{b}{a} = \tan\gamma.$$

We shall develop various laws regarding the variation of the elliptic elements from these simple formulæ, and by way of numerical illustration shall apply them to the case of reflexion from diamond and realgar. These substances are selected, because, for them, we have the careful experiments of Jamin to compare with theory, and so to assure us of the firm foundations on which we are building, and also because while their refractive indices are almost identical, their coefficients of ellipticity at the Principal Incidence differ very considerably, so that their comparison should bring into strong relief the influence of this ellipticity. The ellipticity in the case of realgar is abnormally high; it stands near the head of Jamin's list, while diamond occupies a place near the middle. In realgar we shall take $\mu = 2{\cdot}420$ and $\epsilon_0 = 0{\cdot}1120$, and for diamond $\mu = 2{\cdot}434$ and $\epsilon_0 = 0{\cdot}0242$. This makes $\phi_0 = \tan^{-1}\mu = 67^\circ\, 33'$ in the case of realgar, and $67^\circ\, 40'$ in the case of diamond. The difference of phase (Δ) and the coefficient of ellipticity ($\epsilon = \mathrm{R}/\mathrm{R}'$) are then as follows:—

	$\phi = 0$.	10°.	30°.	60°.	65°.	67° 30′.	67° 40′.
Δ (realgar) ……	0	0° 18′	3° 0′	27° 43′	57° 6′	89° 18′	91° 44′
Δ (diamond) …	0	0° 4′	0° 39′	5° 54′	17° 18′	77° 26′	90°
ϵ (realgar) ……	1	0·9750	0·7829	0·2022	0·1343	0·1120	0·1125
ϵ (diamond) …	1	0·9750	0·7838	0·2052	0·0749	0·0247	0·0242

	$\phi = 0$.	68°.	70°.	75°.	80°.	90°.
Δ (realgar) ……	0	96° 43′	123° 14′	155° 56′	168°	180°
Δ (diamond) …	0	111° 24′	160° 24′	174° 6′	177° 21′	180°
ϵ (realgar) ……	1	0·1140	0·1352	0·2370	0·4329	1
ϵ (diamond) …	1	0·0281	0·0699	0·2348	0·4301	1

These results are represented graphically in figs. 1, 2, 3, and 4 below. In these figures the crosses (×) indicate the results of Jamin's experiments, so that the agreement between theory and experiment is readily estimated.

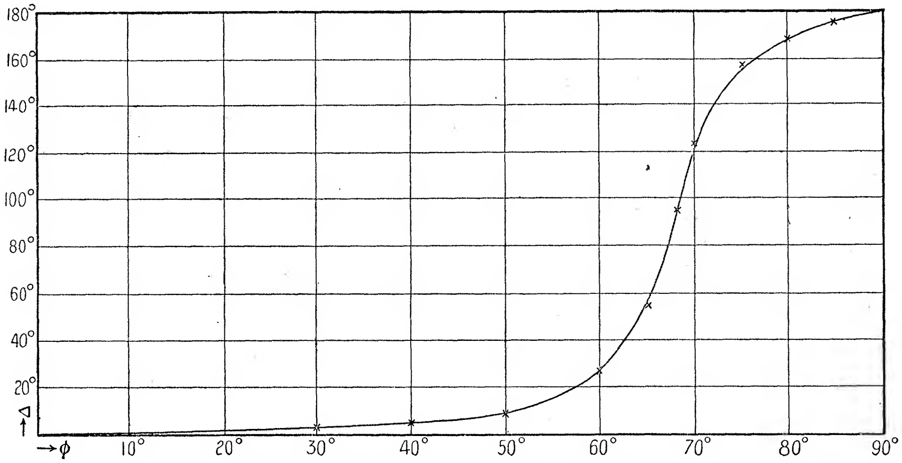

Fig. 1.—Difference of phase for realgar.

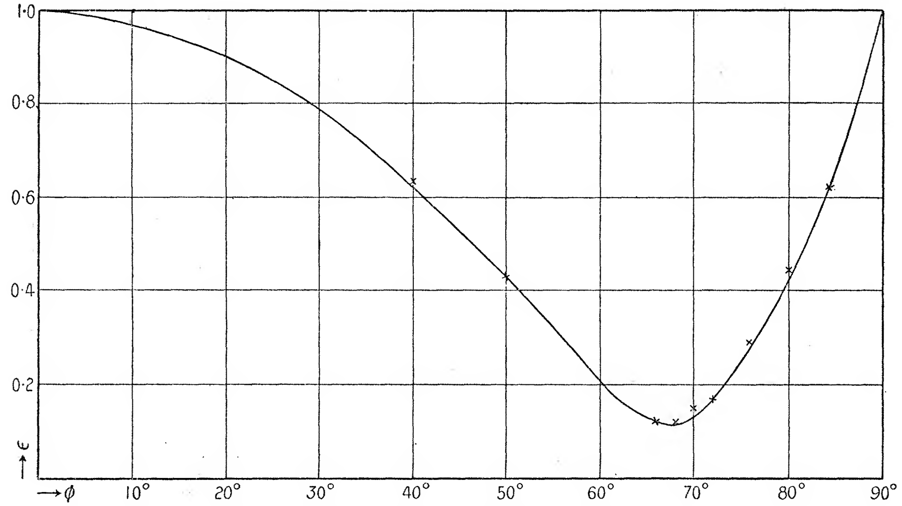

Fig. 2.—Coefficient of ellipticity for realgar.

We have seen that the azimuth of the major axis of the ellipse is given by the formula $\tan 2\psi = \tan 2\alpha \,.\, \cos \Delta$, where $\tan \alpha = \epsilon \tan \theta$.

At direct incidence $\epsilon = 1$ and $\Delta = 0$, so that $\psi = \alpha = \theta$; at grazing incidence $\epsilon = 1$ and $\Delta = \pi$, so that $\psi = -\alpha = -\theta$. Thus as the incidence increases ψ will diminish from θ to $-\theta$, and for most azimuths it will pass through the value zero at the Principal Incidence where $\cos \Delta = 0$.

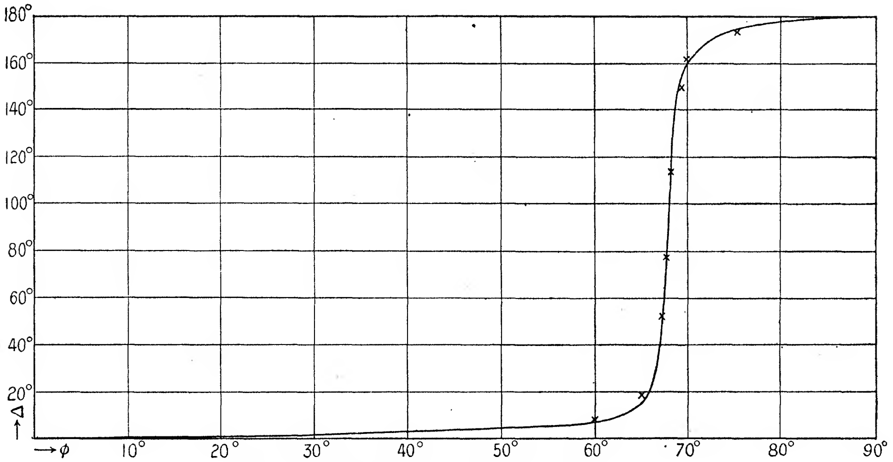

FIG. 3.—Difference of phase for diamond.

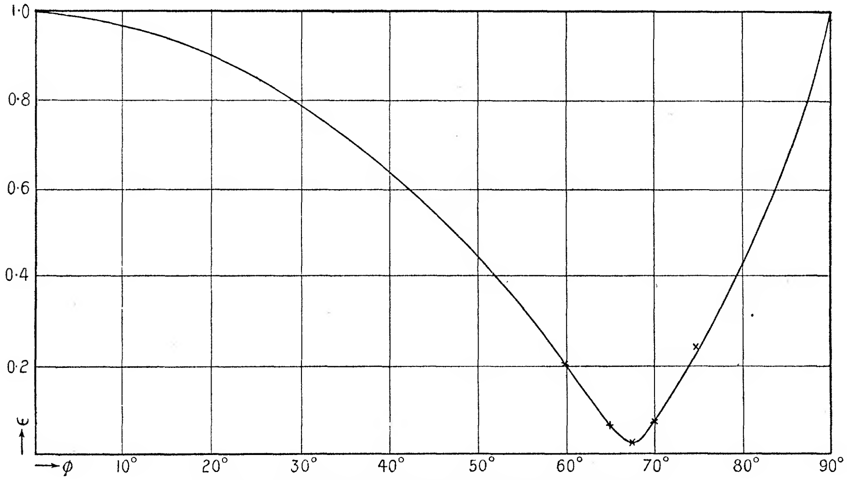

FIG. 4.—Coefficient of ellipticity for diamond.

However, for azimuths greater than $\Omega = 90° - \omega_0$, where $\epsilon_0 = \tan\omega_0$, this will not be the case. In such circumstances $\tan\psi$ will pass through infinity at the Principal Incidence, and ψ will change from $90°$ to $-90°$. Since $\tan 2\psi = \tan 2\alpha \, . \cos\Delta$, it will be seen that if ϕ be less than the Principal Incidence, $\tan 2\psi$ is negative, and ψ is less than $90°$; while if ϕ be greater than the Principal Incidence, ψ is greater than $-90°$. Hence, for such azimuths the graph for ψ will, in the neighbourhood of the Principal Incidence, have the form represented in figs. 5 and 6 below.

There must, therefore, be some incidences for which the azimuth ψ is a minimum or maximum. To find them we note that ψ is stationary when

$\tan\Delta \, . \, \frac{d\Delta}{d\phi} = \frac{\tan\theta}{\tan\alpha \, . \cos 2\alpha} \, . \, \frac{d\epsilon}{d\phi}$. The minima will be found in the neighbourhood of the Principal Incidence (ϕ_0). Putting $\phi = \phi_0 + x$, we have

$$\tan\Delta = \mathrm{A}\frac{\sin\phi\tan\phi}{\tan^2\phi - \mu^2} = \frac{\mathrm{B}}{x}(1 + bx + cx^2 + \ldots);$$

therefore

$$\tan\Delta\,\frac{d\Delta}{d\phi} = -\frac{(1+bx)}{x\,(1+2bx)}.$$

Also $\epsilon = \epsilon_0(1 - 2psx + px^2)$. This makes ϵ a minimum when $x = s$, and, as the minimum is very near ϕ_0, s will be very small.

$$\frac{d\epsilon}{d\phi} = 2p\epsilon_0(-s + x);$$

$$\cos 2\alpha = \frac{1 - \tan^2\alpha}{1 + \tan^2\alpha} = \frac{1 - \tan^2\alpha_0\,(1 - 2psx + px^2)^2}{1 + \tan^2\alpha_0\,(1 - 2psx + px^2)^2} = \cos 2\alpha_0\,(1 + s'x + qx^2),$$

where s' is a small quantity.

Thus the equation for x becomes

$$\frac{1 + bx}{x\,(1 + 2bx)} = \frac{-2p\,(-s + x)}{\cos 2\alpha_0\,(1 + s'x + qx^2)\,(1 - 2psx + px^2)}.$$

p is positive, so that for real (small) values of x we must have $\cos 2\alpha_0$ negative. This requires $\alpha_0 > 45°$ and $\tan\theta > \epsilon_0^{-1}$ or $\theta > 90° - \omega_0 > \Omega$.

It is then only for such azimuths that ψ is stationary anywhere.

From the equation $\tan\alpha_0 = \epsilon_0 \tan\theta$ we see that α_0 increases from 45° to 90° as θ increases from Ω to 90°. Hence in this range $\cos 2\alpha_0$ varies from 0 to -1, and we may put $\cos 2\alpha_0 = -2p\kappa^2$, where κ^2 is small when θ is near Ω. The equation for x then becomes

$$x^2 - sx = \kappa^2(1 - bx)\,(1 + s'x + qx^2)\,(1 - 2psx + px^2). \qquad (1)$$

If x, s, and κ be all of the same order of small quantities, then by retaining only the lowest terms in (1) we get $x^2 - sx = \kappa^2$, so that $x = \frac{1}{2}s \pm \sqrt{(\kappa^2 + \frac{1}{4}s^2)}$. One root is positive and the other negative, the positive root being the larger. Thus the maximum beyond the Principal Incidence is somewhat further from that angle than the minimum that occurs before the Principal Incidence is reached.

If s be so small as to be negligible, then $s = s' = 0$ and (1) becomes $x^2 = \kappa^2(1 - bx)$ on neglecting terms of the fourth order of small quantities.

This gives a positive root, $x = \kappa(1 - \frac{1}{2}b\kappa)$, and a negative root, $x = -\kappa(1 + \frac{1}{2}b\kappa)$. If b be negative, the positive root is numerically the larger; while if b be positive, the negative root is the larger. It will be found that $b = (3 - \mu^2)/2\mu$, so that the sign of b depends on whether μ

is greater or less than $\sqrt{3}$. It will be negative for such highly refringent substances as diamond and realgar, but more usually positive.

If s be not zero, but small compared with x, we get for the positive root of (1) above, $x^2 = \kappa^2(1-b\kappa)+s\kappa$, so that $x = \kappa+\frac{1}{2}s-\frac{1}{2}b\kappa^2$, and for the negative root $x^2 = \kappa^2(1+b\kappa)-s\kappa$ and $x = -\kappa+\frac{1}{2}s-\frac{1}{2}b\kappa^2$.

The following tables give the values of ψ for the azimuth $\theta = 89°$ in the neighbourhood of the Principal Incidence for diamond and realgar. It will be seen that the maxima and minima are much nearer the Principal Incidence in the first case than in the second. The figures that follow represent the same results graphically, fig. 5 referring to diamond and fig. 6 to realgar; the scale of ϕ in the former figure being four times that in the latter :—

Diamond.

ϕ.	65°.	67° 30′.	67° 35′.	67° 39′.	67° 40′.	67° 41′.	67° 45′.	67° 50′.	68°.	70°.
ψ …………	77° 24′	74° 15′	72° 38′	86° 6′	90° −90°	−86° 17′	−70° 54′	−64° 21′	−69° 21′	−76° 41′

Realgar.

ϕ.	60°.	61°.	62°.	63°.	64°.	65°.	66°.	67° 30′.	68°.	69°.	70°.	71°.	72°.	73°.	74°.	75°.
ψ ………	85° 38′	85° 24′	85° 19′	85° 12′	85° 24′	85° 54′	86° 56′	89° 53′	−88° 57′	−87° 2′	−85° 54′	−85° 28′	−85° 24′	−85° 34′	−85° 52′	−86° 9

From these tables and figures it appears that, in the case of diamond, the maximum value of ψ occurs about 10 minutes beyond the Principal Incidence and the minimum about 5 minutes below that angle; while the corresponding quantities for realgar are 5 degrees and $4\frac{1}{2}$ degrees respectively.

For azimuths less than Ω we have seen that there are no such maxima and minima, and that ψ decreases throughout from θ to $-\theta$ as the incidence increases, and passes through zero at the Principal Incidence. The following

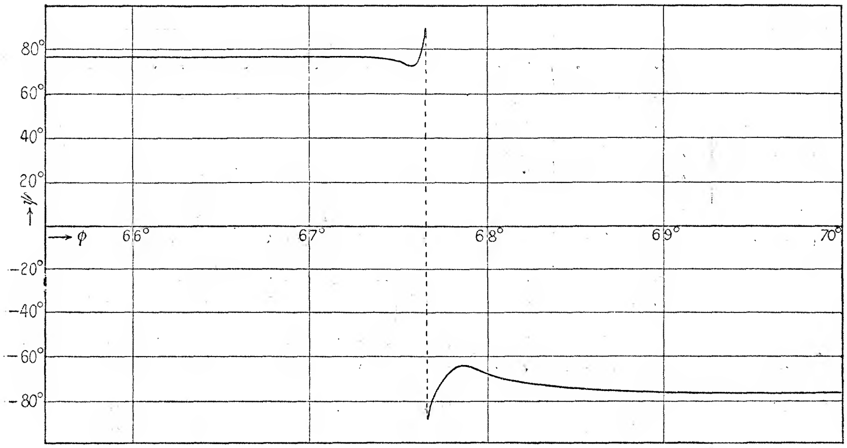

Fig. 5.—Diamond.

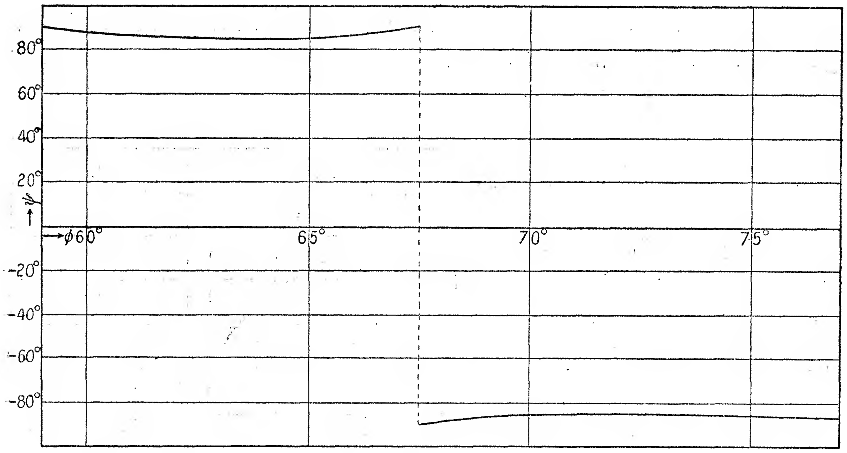

Fig. 6.—Realgar.

table sets out the values of ψ for various azimuths (θ) and incidences (ϕ) in the case of diamond and realgar. The first entry in each case refers to realgar and the second to diamond.

θ.	ϕ = 10°.	30°.	60°.	65°.	67° 30′.	67° 40′.	68°.	70°.	75°.	80°.
°	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′	° ′
10	9 45 9 45	7 52 7 52	1 49 2 3	0 44 0 44	0 1 0 3	− 0 2 0	− 0 8 − 0 6	− 0 45 − 0 40	− 2 12 − 2 21	− 4 16 − 4 20
30	29 22 29 23	24 19 24 21	5 54 6 45	2 25 2 22	0 3 0 11	− 0 7 0	− 0 27 − 0 19	− 2 28 − 2 11	− 7 8 − 7 41	−13 46 −13 56
45	44 16 44 16	38 4 38 5	10 18 11 30	4 14 4 6	0 5 0 19	− 0 12 0	− 0 46 − 0 33	− 4 18 − 3 46	−12 19 −13 9	−23 6 −23 15
60	59 22 59 22	53 36 53 37	17 38 19 30	7 29 7 5	0 8 0 32	− 0 21 0	− 1 23 − 0 57	− 7 36 − 6 31	−21 1 −22 3	−36 42 −36 40
75	74 38 74 17	71 6 71 8	36 3 37 25	18 0 15 2	0 21 1 10	− 0 53 0	− 3 28 − 2 3	−18 27 −13 32	−41 10 −41 13	−58 27 −58 6
85	84 53 79 45	83 37 77 19	68 21 67 0	64 33 40 22	88 35 3 48	−86 36 0	−78 11 − 6 45	−64 41 −38 14	−71 0 −69 42	−78 48 −78 31
89	88 58 88 59	88 44 88 43	85 38 85 9	85 54 77 24	89 53 74 15	−89 43 90 0	−88 57 −69 21	−85 54 −76 41	−86 9 −85 46	−87 44 −87 40

These results are exhibited in figs. 7 and 8 below, the first dealing with realgar and the second with diamond.

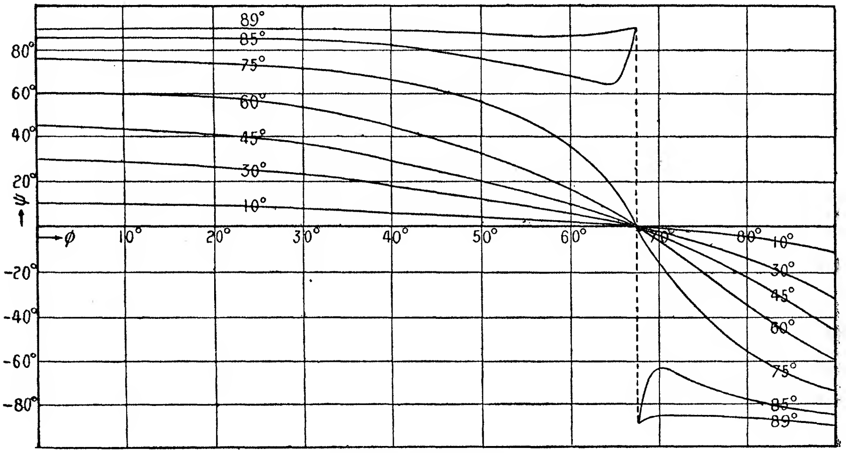

Fig. 7.—Realgar.

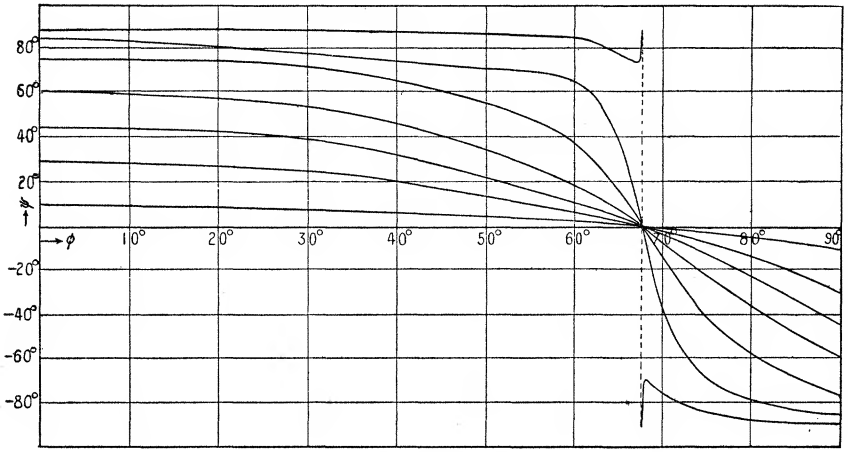

FIG. 8.—Diamond.

We have seen that if the transition from one medium to another were abrupt instead of gradual, the reflected light would always be plane polarised. The azimuth of the plane of polarisation would be α or $-\alpha$, according as the angle of incidence were less or greater than the Principal Incidence. The angle α is given by the equation $\tan\alpha = \epsilon . \tan\theta$, and it must be remembered that ϵ is not the same in the neighbourhood of the Principal Incidence in the two cases of abrupt and gradual transition. Fig. 9 represents an ellipse whose centre is O and of which OA is the semi-major axis. OI and OP are parallel and perpendicular respectively to the plane of incidence. EH and FH are tangents parallel to OP and OI respectively. OE represents the amplitude of the displacement parallel to the plane of incidence, and OF the corresponding

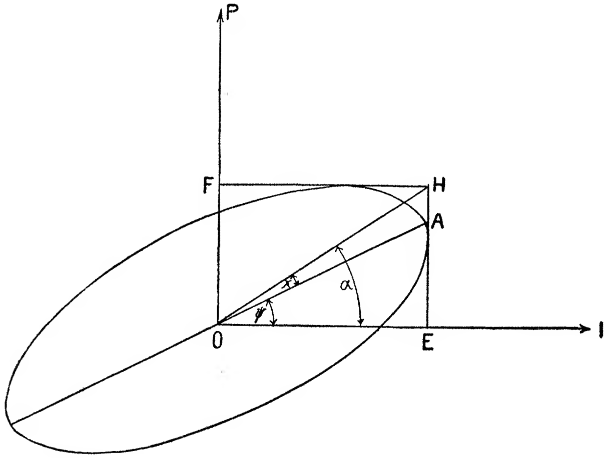

FIG. 9.

amplitude for the perpendicular plane. Thus OE $=$ R$'\cos\theta$ and OF $=$ R$\sin\theta$, $\tan$ HOE $=$ OF/OE $= \epsilon\tan\theta$, so that HOE represents the angle α. It will be seen from such a figure, or from the equation $\tan 2\psi = \cos\Delta \times \tan 2\alpha$, that ψ is numerically less or greater than α, according as α is less or greater than 45°. Let χ represent the amount of turning *towards* the plane of incidence in passing from the azimuth of plane polarised light in the case of abrupt transition to the axis of the elliptic orbit when the layer of transition is taken into account. χ will be positive or negative according as α is less or greater than 45°, *i.e.*, according as $\epsilon\tan\theta$ is less or greater than unity. Thus χ will be negative for all angles of incidence when the azimuth is greater than Ω. It will also be negative for high and low incidences, when ϵ is nearly unity, for azimuths greater than 45°; in all other cases it will be positive. At the Principal Incidence, χ is zero for azimuths less than Ω, and 90° for greater azimuths. Thus the layer of transition not only replaces linear by elliptical polarisation, but, when the azimuth exceeds Ω, it turns the direction of the "major axis" from the plane of incidence through a right angle.* It will be seen later that the polarisation is circular at the azimuth Ω, so that there is no discontinuity in passing through that angle. The following table sets out the values of χ for various azimuths and incidences, the first entry referring to realgar and the second to diamond:—

θ.	φ = 10°.	30°.	60°.	65°.	67° 30′.	67° 40′.	68°.	70°.	75°.	80°.
° 10	0 0	0 0	0 ·12 0 ·1	0 0 ·2	0 0	0 0	0 0	0 0 ·2	0 ·12 0 ·1	0 ·6 0
30	0 0	0 0	0 ·45 0 ·2	0 ·2 0 ·7	0 ·1 0 ·1	0 ·1 0	0 ·1 0 ·1	0 ·3 0 ·8	0 ·39 0 ·2	0 ·16 0 ·1
45	0 0	0 0	1 ·12 0 ·6	0 ·8 0 ·11	0 ·1 0 ·2	0 0	0 ·1 0 ·1	0 ·7 0 ·14	1 ·1 0 ·4	0 ·19 0 ·1
60	0 0	0 0	1 ·40 0 ·5	0 ·25 0 ·19	0 ·1 0 ·2	0 ·2 0	0 ·4 0 ·3	0 ·23 0 ·23	1 ·18 0 ·5	0 ·10 0 ·1
75	0 0	0 0	0 ·58 0 ·2	0 ·23 0 ·35	0 ·5 0 ·6	0 ·11 0	0 ·38 0 ·5	1 ·12 1 ·5	0 ·20 0 ·1	−0 ·13 0 ·1
85	0 0	0 0	−1 ·45 0 ·5	−25 ·17 0 ·12	−87 ·47 0 ·34	−84 ·26 0	−69 ·34 0 ·15	−24 ·48 0 ·23	−1 ·16 0 ·8	−0 ·13 0 ·1
89	0 0	0 0	−0 ·33 −0 ·1	− 9 ·36 − 0 ·31	−85 ·50 −58 ·26	−79 ·1 −90 ·0	−51 ·42 −39 ·36	− 9 ·24 − 0 ·42	−0 ·22 −0 ·1	−0 ·3 0

We shall next consider the variation of the quantity b/a, the ratio of the axes, on which the eccentricity of the ellipse depends. We have $b/a = \tan\gamma$, where $\sin 2\gamma = \sin \Delta \sin 2\alpha$ and $\tan \alpha = \epsilon . \tan \theta$. Thus b/a vanishes when $\theta = 0$ or $\frac{1}{2}\pi$, or when $\Delta = 0$ or π. These, of course, are the cases of plane polarisation, $\theta = 0$ or $\frac{1}{2}\pi$ being azimuths in or perpendicular to the plane of incidence, and $\Delta = 0$ or π being direct or grazing incidence at any azimuth.

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* See figs. 21 and 22 below.

For any given azimuth b/a begins at zero for direct incidence, and ends at zero for grazing incidence. It must pass through certain maxima or minima in this interval. From the equation for γ we see that b/a is stationary when $\cos \Delta . d\Delta \sin 2\alpha + 2 \sin \Delta \cos 2\alpha . d\alpha = 0$.

This is satisfied at the Principal Incidence where $\cos \Delta = 0$, and $d\alpha = 0$, since α is stationary with ϵ. As we move in either direction from the Principal Incidence, $\sin \Delta$ diminishes and, for all azimuths less than Ω, $\sin 2\alpha$ increases. For substances such as diamond, for which the variation of Δ is *very* rapid in the neighbourhood of the Principal Incidence, the variation of γ will be mainly controlled by that of Δ, and we shall have b/a a maximum at the Principal Incidence. With other substances, such as realgar, for which the variation of Δ is less rapid, the increase in $\sin 2\alpha$ may more than counterbalance the decrease in $\sin \Delta$. In such circumstances b/a will be a minimum at the Principal Incidence, and there will be two maxima on each side of this and not far from it. Especially with substances of high refractive index, the increase of ϵ beyond the Principal Incidence is much more rapid than its decrease before, so that the maximum beyond the Principal Incidence will be further from it than the one below that incidence. For azimuths greater than Ω, we have $2\alpha > 90°$, so that $\sin 2\alpha$ diminishes with the increase of ϕ. In this case, therefore, b/a diminishes as the incidence increases beyond the Principal Incidence. There can thus be no maximum beyond that incidence, the only maximum possible being one very near the Principal Incidence and below it.

The following tables set out the values of b/a and of the eccentricity (e) for realgar and diamond, and the results for b/a are exhibited graphically in figs. 10 and 11 :—

θ.		$\phi = 10°$.	30°.	60°.	65°.	67° 30′.	67° 40′.	68°.	70°.	75°.	80°.
Realgar.											
10°	b/a	0·00087	0·00698	0·01629	0·01978	0·01978	0·01978	0·01993	0·01978	0·01716	0·01571
	e	0·99999	0·99998	0·99989	0·99980	0·99980	0·99980	0·99978	0·99980	0·99984	0·99987
30°	b/a	0·00233	0·01949	0·05387	0·06496	0·06467	0·06496	0·06525	0·06496	0·05474	0·04891
	e	0·99999	0·99982	0·99856	0·99790	0·99793	0·99790	0·99787	0·99790	0·99851	0·99881
45°	b/a	0·00262	0·02531	0·09159	0·11217	0·11187	0·11217	0·11305	0·11246	0·09218	0·07636
	e	0·99999	0·99968	0·99580	0·99369	0·99374	0·99369	0·99475	0·99367	0·99575	0·99710
60°	b/a	0·00233	0·02502	0·14826	0·19227	0·19378	0·19438	0·19589	0·21073	0·14618	0·10069
	e	0·99999	0·997	0·9891	0·9815	0·9811	0·9811	0·9809	0·9774	0·9895	0·9949
75°	b/a	0·00145	0·01600	0·23608	0·38653	0·41728	0·41923	0·42105	0·38888	0·21134	0·09423
	e	0·99999	0·9999	0·9718	0·9221	0·9088	0·9080	0·9069	0·9210	0·9774	0·9956
85°	b/a	0·00029	0·00582	0·17453	0·46843	0·78175	0·77615	0·74810	0·46348	0·13491	0·04046
	e	1	0·99999	0·9847	0·8837	0·6236	0·6306	0·6636	0·8861	0·9908	0·9993
89°	b/a	0	0·00116	0·03988	0·10863	0·15600	0·15511	0·15183	0·10775	0·02997	0·00843
	e	1	0·99999	0·9995	0·9917	0·9879	0·9879	0·9881	0·9931	0·9995	0·99996
Diamond.											
10°	b/a	0·00029	0·00145	0·00378	0·00407	0·00436	0·00436	0·00435	0·00435	0·00430	0·00349
	e	1	0·99997	0·99997	0·99997	0·99997	0·99997	0·99997	0·99997	0·99997	0·99997
30°	b/a	0·00036	0·00436	0·01193	0·01280	0·01396	0·01396	0·01396	0·01367	0·01360	0·01076
	e	1	0·99999	0·99993	0·99992	0·99990	0·99990	0·99990	0·99990	0·99991	0·99994
45°	b/a	0·00058	0·00553	0·02037	0·02211	0·02415	0·02444	0·02442	0·02328	0·02298	0·01687
	e	1	0·99996	0·99981	0·99976	0·99971	0·99970	0·99970	0·99973	0·99974	0·99988
60°	b/a	0·00038	0·00553	0·03259	0·03812	0·04191	0·04220	0·04220	0·04016	0·03579	0·02211
	e	1	0·99996	0·99948	0·99928	0·99913	0·99912	0·99912	0·99920	0·99935	0·99976
75°	b/a	0·00029	0·00349	0·04978	0·07753	0·09013	0·09103	0·09101	0·08251	0·05124	0·02066
	e	1	0·99999	0·99876	0·99699	0·99598	0·99598	0·99600	0·99659	0·99870	0·99956
85°	b/a	0·00029	0·00233	0·03725	0·15034	0·27482	0·27607	0·27513	0·16824	0·03346	0·00902
	e	1	0·99999	0·99932	0·98863	0·96141	0·96114	0·96132	0·98576	0·99943	0·99996
89°	b/a	0	0·00029	0·00873	0·06584	0·66063	0·72255	0·56885	0·07929	0·00756	0·00175
	e	1	1	0·99996	0·99781	0·75071	0·69132	0·82264	0·99685	0·99997	0·99999

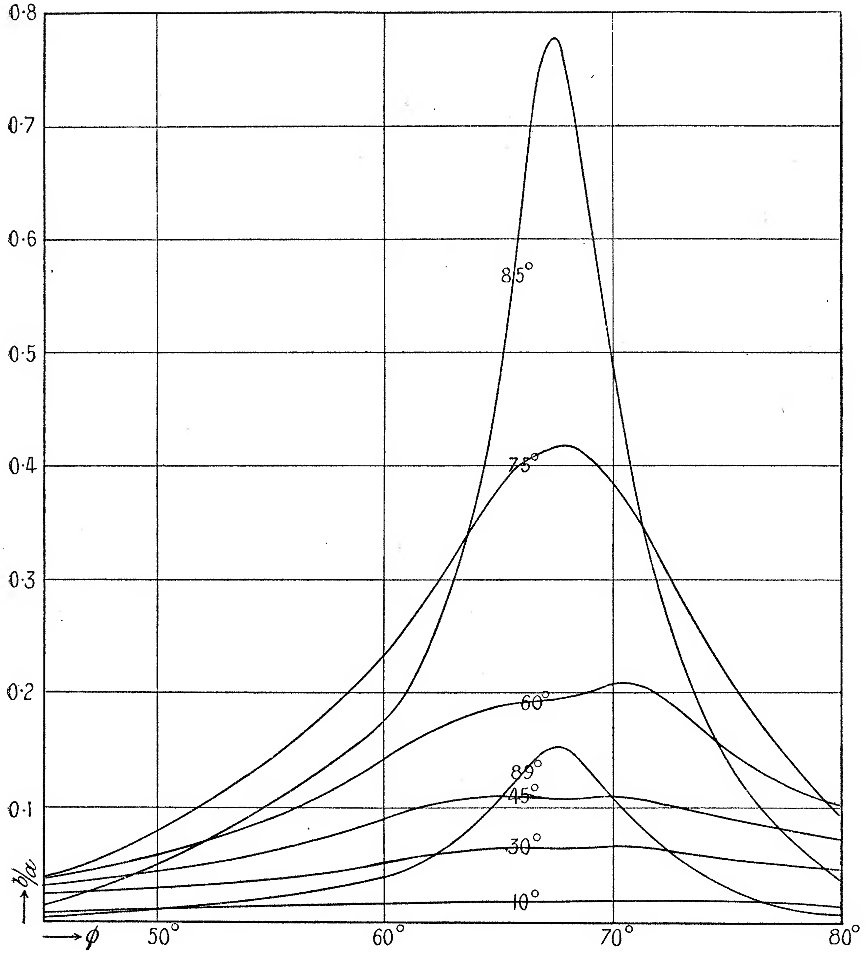

FIG. 10.—Realgar.

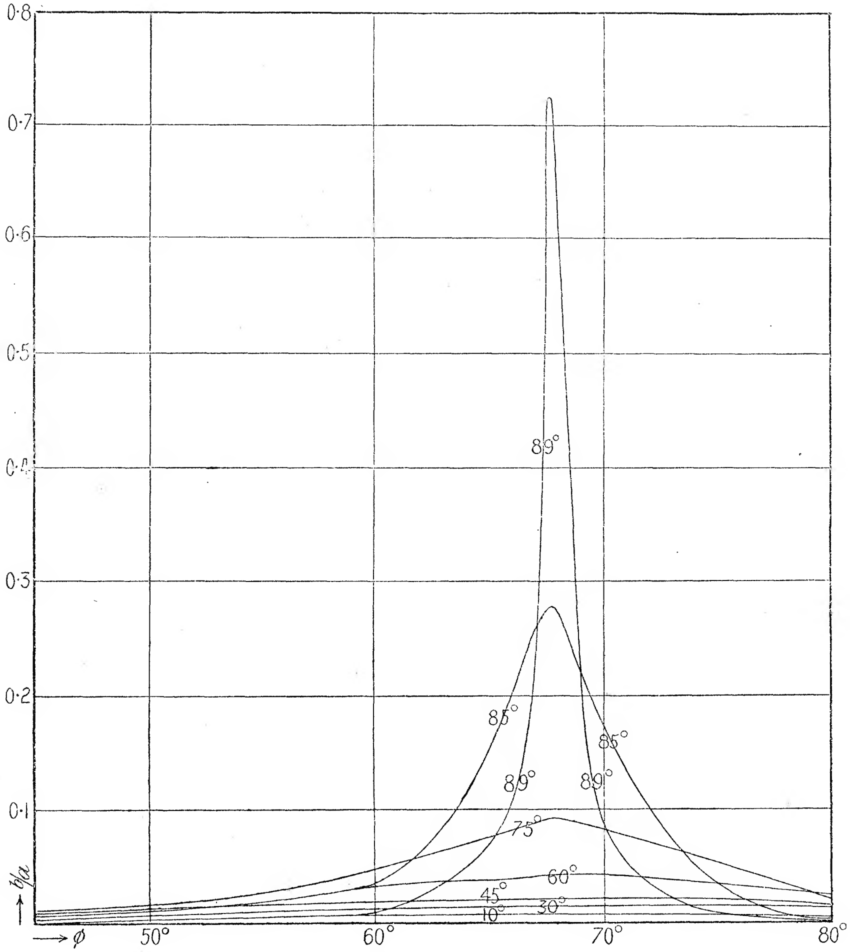

Fig. 11.—Diamond.

On p. 493 are considered the variation of b/a for different incidences at a given azimuth. Its variation for a given incidence at different azimuths is very simply discussed. Under such circumstances ϵ and Δ are given, so that $\sin 2\gamma$ varies as $\sin 2\alpha$, and γ is greatest when $\alpha = 45°$, in which case $\gamma = \frac{1}{2}\Delta$ or $\frac{1}{2}(\pi - \Delta)$, according as the incidence is below or beyond the Principal Incidence. Thus at a given incidence b/a begins at zero with $\theta = 0$, and ends with zero at $\theta = 90°$. It passes through a maximum at the azimuth $\theta = \cot^{-1}\epsilon$, and its maximum value is $\tan \frac{1}{2}\Delta$ or $\cot \frac{1}{2}\Delta$, according as the incidence is less or greater than the Principal Incidence. The following table gives the position and magnitude of the maxima for various incidences:—

ϕ.	10°.	30°.	60°.	65°.	67° 30′.
θ (realgar) ………	45° 44′	51° 56′	78° 30′	82° 21′	83° 37′
b/a (realgar) ……	0 ·00262	0 ·02619	0 ·24686	0 ·54409	0 ·98786
θ (diamond) ……	45° 42′	51° 55′	78° 24′	85° 43′	88° 35′
b/a (diamond) …	0 ·00058	0 ·00567	0 ·05153	0 ·15213	0 ·80163

ϕ.	67° 40′.	68°.	70°.	75°.	80°.
θ (realgar) ………	83° 35′	83° 30′	82° 18′	76° 40′	66° 35′
b/a (realgar) ……	0 ·97020	0 ·88888	0 ·54032	0 ·21316	0 ·10510
θ (diamond) ……	88° 37′	88° 30′	85°	76° 47′	66° 43′
b/a (diamond) …	1	0 ·68215	0 ·17273	0 ·05153	0 ·02328

These results are represented in figs. 12 and 13 below, the first representing the plan (θ) and the second the elevation (b/a).

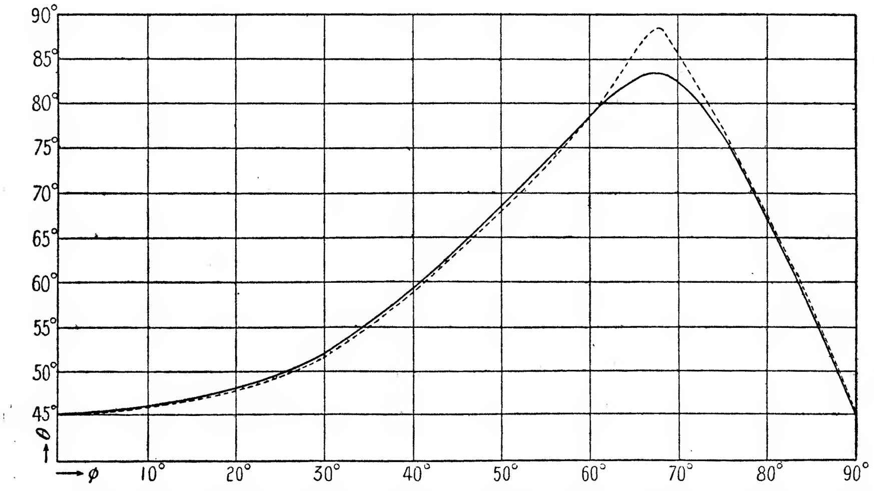

Fig. 12.—The continuous curve represents the results for realgar, the dotted one for diamond.

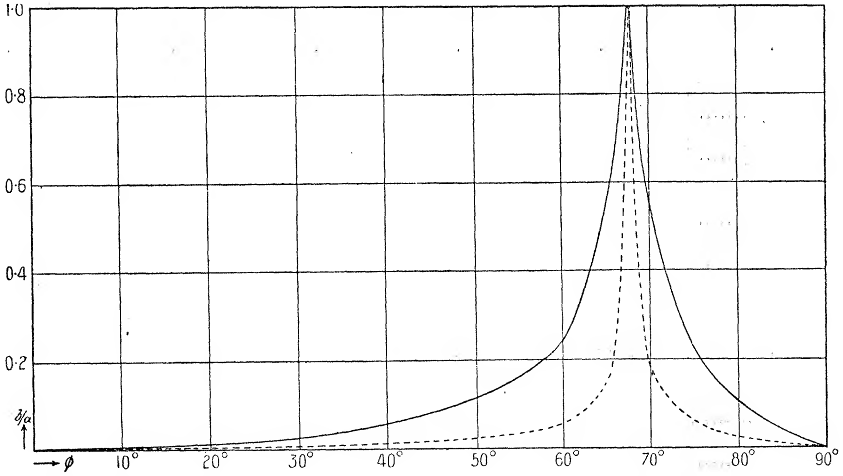

FIG. 13.—The continuous curve corresponds to realgar, the dotted curve to diamond.

The axes of the elliptic orbit are given by the equations

$$a = \frac{\mathrm{R}\sin\theta\cos\gamma}{\sin\alpha} = \frac{\mathrm{R}'\cos\theta\cos\gamma}{\cos\alpha}$$

and

$$b = a\tan\gamma = \frac{\mathrm{R}\sin\theta\sin\gamma}{\sin\alpha} = \frac{\mathrm{R}'\cos\theta\sin\gamma}{\cos\alpha}.$$

At direct or grazing incidence we have $\gamma = 0$ and $\alpha = \theta$, so that $a = \mathrm{R} = \mathrm{R}'$ for all azimuths. We shall consider how a varies with the incidence and azimuth. We have

$$a^2 = \frac{ab}{\tan\gamma} = \frac{\mathrm{R}^2\sin\Delta}{2\epsilon}\,\frac{\sin 2\theta}{\tan\gamma}.$$

Thus for a given incidence $a^2 = \kappa\,.\,\dfrac{\sin 2\theta}{\tan\gamma}$, where κ is a constant. Hence we have

$$\begin{aligned}\frac{2a\,da}{\kappa} &= \cot^2\gamma\,[2\tan\gamma\cos 2\theta\,d\theta - \sec^2\gamma\sin 2\theta\,d\gamma] \\ &= \cot\gamma\,.\,d\theta\left[\cos 2\theta - \frac{\cos 2\alpha}{\cos 2\gamma}\right].\end{aligned}$$

Since $\tan\alpha = \epsilon\,.\tan\theta$, we have $\alpha < \theta$ and $\cos 2\alpha > \cos 2\theta$, 2α and 2θ both being less than π. Moreover, 2γ is never greater than $\frac{1}{2}\pi$, so that $\cos 2\gamma$ is positive and less than unity. Hence if 2α be less than $\frac{1}{2}\pi$, so that $\cos 2\alpha$ is positive, we have $\dfrac{\cos 2\alpha}{\cos 2\gamma} > \cos 2\alpha > \cos 2\theta$; $\cos 2\theta - \dfrac{\cos 2\alpha}{\cos 2\gamma}$ is consequently negative, and a diminishes as θ increases. However, this argument is no longer good when 2α is greater than 90°. We shall, therefore, proceed to investigate the roots of the equation $\cos 2\theta = \cos 2\alpha/\cos 2\gamma$. Combining this with $\sin 2\gamma = \sin\Delta\,.\,\sin 2\alpha$, we get—

$$\frac{\cos 2\alpha}{\cos 2\theta} = \cos 2\gamma = +(1-\sin^2\Delta\sin^2 2\alpha)^{\frac{1}{2}},$$

the positive sign being taken, as $\cos 2\gamma$ is positive. For brevity, write $x = \tan^2\alpha$, $s = \sin^2\Delta$, $c = \epsilon^2$, so that we have

$$\frac{(1-x)(c+x)}{(1+x)(c-x)} = +\left(1-\frac{4sx}{(1+x)^2}\right)^{\frac{1}{2}},$$

an equation obviously satisfied by $x = 0$, corresponding to $\alpha = 0 = \theta$.

Squaring, and clearing of fractions and of the root $x = 0$, we get

$$x^2(1-c-s)+2scx-c+c^2(1-s) = 0;$$

$$x(1-c-s) = -sc\pm(1-c)\{c(1-s)\}^{\frac{1}{2}}.$$

The process of squaring introduces an irrelevant root. To exclude this the sign of $\frac{1-x}{1+x}\cdot\frac{c+x}{c-x}$ is to be positive, and as $\frac{c+x}{1+x}$ is necessarily positive this requires $\frac{1-x}{c-x}$ to be positive. The value of x given above makes

$$\frac{1-x}{c-x} = \mp\left(\frac{1-s}{c}\right)^{\frac{1}{2}} = \mp\frac{\cos\Delta}{\epsilon},$$

and as $\cos\Delta$ is positive or negative according as the incidence is less or greater than the Principal Incidence, we must take the lower sign below the Principal Incidence and the upper sign beyond it.

Thus below the Principal Incidence we have

$$\tan^2\theta = \frac{\tan^2 d}{\epsilon^2} = \frac{x}{\epsilon^2} = -\frac{1-\epsilon\cos\Delta}{\epsilon(\cos\Delta-\epsilon)}.$$

But $1-\epsilon\cos\Delta$ is positive, and $\cos\Delta-\epsilon$ is also positive, except very near the Principal Incidence. Hence, except in the immediate neighbourhood of the Principal Incidence, $\tan^2\theta$ would be negative if a were stationary, so that there is no real value of θ for which a is stationary. Beyond the Principal Incidence our formula gives

$$\tan^2\theta = \frac{1+\epsilon\cos\Delta}{\epsilon(\cos\Delta+\epsilon)}.$$

The numerator of this fraction is positive and the denominator negative, except very near the Principal Incidence. We see then that da is negative throughout, except in the immediate vicinity of the Principal Incidence, so that a diminishes as the azimuth increases for all incidences outside a small region near the Principal Incidence. This region extends on each side of the Principal Incidence from $\cos\Delta = \epsilon$ to $\cos\Delta = -\epsilon$. Suppose that

$\cos\Delta = \pm\epsilon$ when $\phi = \phi_0 + x$, where ϕ_0 is the Principal Incidence and x is small. We have

$$\tan\Delta = -4\epsilon_0 \frac{\mu^2}{\sqrt{(1+\mu^2)}} \frac{\sin\phi\tan\phi}{\tan^2\phi - \mu^2} = -2\epsilon_0 \left(\frac{\mu^2}{1+\mu^2}\right)^2 \left(\tfrac{1}{2} + \frac{2+\mu^2}{\mu}\right),$$

so that

$$\cos\Delta = \cot\Delta\,(\text{approx.}) = -\frac{x}{2\epsilon_0} \Big/ \left(\frac{\mu^2}{1+\mu^2}\right)^2 = \pm\epsilon = \pm\epsilon_0 (1 + px + qx^2),$$

where, as we have seen, p is very small.

This equation gives, very approximately

$$x = \pm 2\epsilon_0{}^2 \left(\frac{\mu^2}{1+\mu^2}\right)^2,$$

and owing to the smallness of $\epsilon_0{}^2$ this is usually very small. For diamond we have $x = \pm 2' 55''$ and for realgar $x = \pm 1° 2' 55''$.

At the Principal Incidence a is stationary when $x = 1$, *i.e.*, when $\cot\theta = \epsilon_0$ and $\theta = \Omega$, the azimuth giving circular polarisation. The formulæ for a and b are

$$a = \frac{R\sin\theta\cos\gamma}{\sin\alpha} = \frac{R'\cos\theta\cos\gamma}{\cos\alpha} \quad\text{and}\quad b = \frac{R\sin\theta\sin\gamma}{\sin\alpha} = \frac{R'\cos\theta\sin\gamma}{\cos\alpha},$$

where $\sin 2\gamma = \sin\Delta \sin 2\alpha$.

At the Principal Incidence $\sin\Delta = 1$, so that $\gamma = \alpha$ or $\frac{1}{2}\pi - \alpha$. If $\gamma = \alpha$ we have $a = R'\cos\theta$ and $b = R\sin\theta$, and we must interchange a and b if we take $\gamma = \frac{1}{2}\pi - \alpha$. As a is the greater of the two semi-axes, the first or second of these values must be assigned to γ, according as $R'\cos\theta$ is greater or less than $R\sin\theta$, *i.e.*, according as θ is less or greater than Ω.

Thus, at the Principal Incidence, if θ be less than Ω we have $a = R'\cos\theta$, while if θ be greater than Ω we have $a = R\sin\theta$. Hence, after passing the azimuth Ω, a increases from $R_0(1+\epsilon_0{}^2)^{-\frac{1}{2}}$ to R_0.

It is, therefore, a minimum at the azimuth Ω. It must be remembered, however, that in passing through this azimuth, the angle ψ which the major axis makes with the plane of incidence suddenly changes from 0° to 90°. This means, of course, that the longer axis is perpendicular to the plane of incidence, instead of being in that plane. At the Principal Incidence the axis in the plane of incidence is $R_0'\cos\theta$ throughout, while the axis perpendicular to that plane is $R_0\sin\theta$. As the azimuth increases, the first axis diminishes steadily from R_0' to zero, while the second increases steadily from zero to R_0.

We have still to consider the variations of a with the incidence for a given azimuth. From the formula for a, viz., $a = R\sin\theta\cos\gamma/\sin\alpha = R'\cos\theta\cos\gamma/\cos\alpha$, we see that when $\theta = 0$, $a = R'$, and when $\theta = 90°$, $a = R$. Thus as the azimuth increases from one of these values to the other, the curve for a must pass from that representing R′, which rises steadily as the angle of incidence increases, to that representing R, which is equal to R′ at direct and grazing incidence, but falls to a minimum in the interval at the Principal Incidence.

All these statements as to the variation of a are illustrated in the table and figures that follow. The table gives the values of a for various incidences and azimuths, the first entry referring to realgar, and the second to diamond. Figs. 14 and 15 below represent the same results graphically.

θ.	ϕ=10°.	30°.	60°.	65°.	67° 30′.	67° 40′.	68°.	70°.	75°.	80°.
° 10	0·4201 0·4218	0·4613 0·4641	0·6256 0·6328	0·6761 0·6671	0·6950 0·6984	0·7004 0·6834	0·7028 0·6982	0·7224 0·7310	0·7756 0·7823	0·8435 0·8453
30	0·4196 0·4194	0·4421 0·4541	0·5558 0·5562	0·5921 0·5932	0·6122 0·6156	0·6144 0·6149	0·6170 0·6186	0·6359 0·6362	0·6898 0·6926	0·7619 0·7628
45	0·4152 0·4169	0·4171 0·4196	0·4552 0·4620	0·4842 0·4871	0·5005 0·5018	0·5018 0·5028	0·5042 0·5054	0·5205 0·5212	0·5712 0·5753	0·6561 0·6580
60	0·4124 0·4141	0·3912 0·3937	0·3576 0·3394	0·3546 0·3453	0·3538 0·3551	0·3550 0·3547	0·3565 0·3569	0·3688 0·3703	0·4221 0·4274	0·5313 0·5330
75	0·4104 0·4128	0·3712 0·3737	0·2012 0·2084	0·1845 0·1834	0·1832 0·1841	0·1837 0·1841	0·1848 0·1849	0·1979 0·1962	0·2668 0·2722	0·4183 0·4186
85	0·4098 0·4125	0·3645 0·3717	0·1378 0·1422	0·0987 0·0779	0·0789 0·0619	0·0796 0·0619	0·0817 0·0627	0·1069 0·0809	0·1968 0·1976	0·3756 0·3739
89	0·4098 0·4113	0·3637 0·3659	0·1292 0·1319	0·0918 0·0527	0·0792 0·0179	0·0798 0·0172	0·0818 0·0197	0·0995 0·0528	0·1875 0·1865	0·3667 0·3681

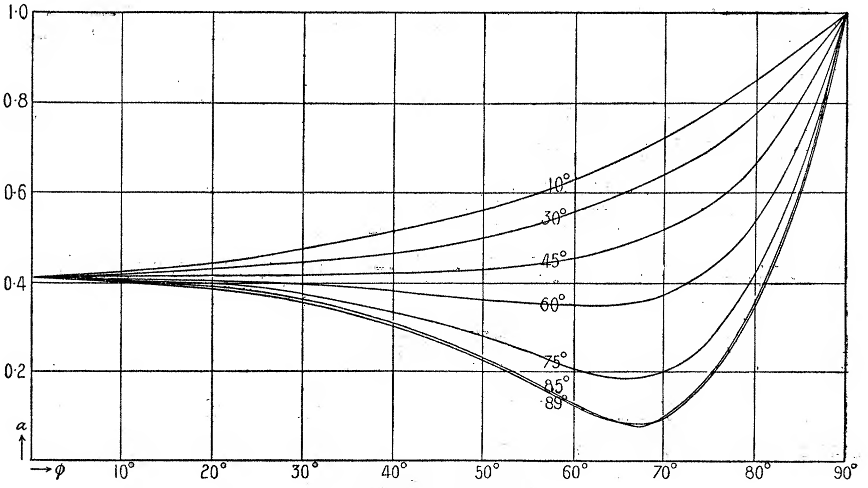

Fig. 14.—Realgar.

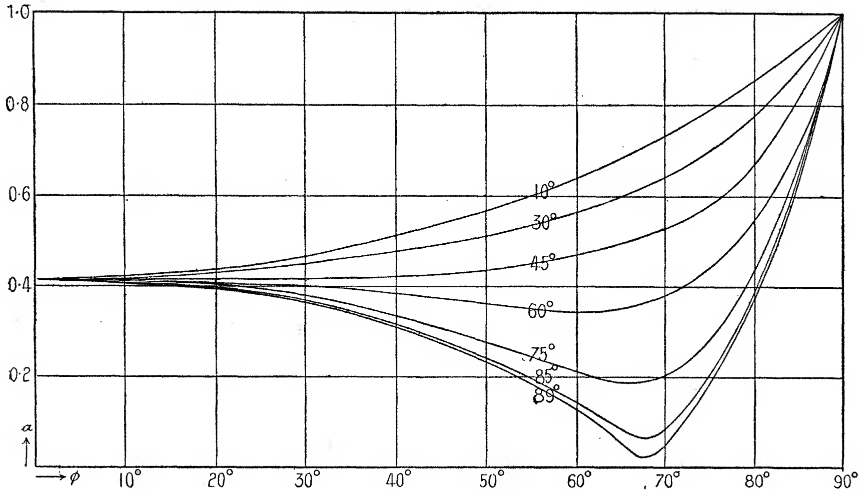

Fig. 15.—Diamond.

We have next to consider the mode of variation of the other semi-axis b, with the incidence and azimuth. The formulæ already obtained give

$$b^2 = a^2\tan^2\gamma = (2\epsilon)^{-1}\mathrm{R}^2\sin\Delta\sin 2\theta\tan\gamma = \kappa'\sin 2\theta\tan\gamma,$$

where κ' is a constant for a given incidence. This enables us to discuss the variation of b with the azimuth. We see that b vanishes with γ when $\theta = 0$ and $\theta = 90°$, so that we should expect it to pass through a maximum somewhere in the interval. The above equation for b^2 gives

$$\frac{2b\,db}{\kappa'} = 2\cos 2\theta\tan\gamma . d\theta + \frac{\sin 2\theta}{\cos^2\gamma} . d\gamma = \frac{\sin 2\gamma . d\theta}{\cos^2\gamma}\left[\cos 2\theta + \frac{\cos 2\alpha}{\cos 2\gamma}\right].$$

Thus b is a maximum when $\cos 2\theta = -\cos 2\alpha/\cos 2\gamma$. Except for the change of sign, this is the same equation as that dealt with above on p. 499. It corresponds, in the notation of that page, with the equation

$$\frac{(1-x)(c+x)}{(1+x)(c-x)} = -\left\{1 - \frac{4sx}{(1+x)^2}\right\}^{\frac{1}{2}}.$$

In the present case we require $1 - x/c - x$ to be negative, and, as before, we have

$$\frac{1-x}{c-x} = \mp\frac{\cos\Delta}{\epsilon}.$$

Hence below the Principle Incidence we must take the upper sign, and the lower sign beyond it. We thus obtain

$$\tan^2\theta = \frac{1 \pm \epsilon\cos\Delta}{\epsilon(\epsilon \pm \cos\Delta)},$$

the upper or lower sign being taken according as the incidence is less or greater than the Principle Incidence. In both cases we get a real value for θ for any given incidence. The magnitude of the corresponding maximum is

$$b = \mathrm{R} \sin \Delta \, (1 \pm 2\epsilon \cos \Delta + \epsilon^2)^{-\frac{1}{2}}.$$

A simple geometrical construction for $\tan^2\theta$ and the maximum of b is easily obtained. In fig. 16, take AB = 1, AC = ϵ, and the angle BAC equal

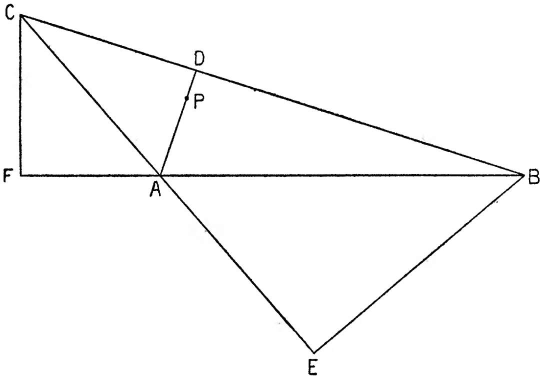

FIG. 16.

to $\pi - \Delta$ or Δ, according as the incidence is less or greater than the Principal Incidence. Draw AD perpendicular to BC, and in it take a point P so that AP = R′ × AD.

Then AP represents the maximum value of b, and BD/CD represents $\tan^2\theta$. To prove this, draw BE and CF perpendicular to CA and AB respectively. Then

$$\sin \Delta \, (1 \pm 2\epsilon \cos \Delta + \epsilon^2)^{-\frac{1}{2}} = \mathrm{BE}/\mathrm{CE} = \sin \mathrm{C} \quad \text{and} \quad \mathrm{R} = \epsilon \,.\, \mathrm{R}' = \mathrm{R}' \,.\, \mathrm{AC}.$$

Hence

$$b = \mathrm{R} \sin \Delta \, (1 \pm 2\epsilon \cos \Delta + \epsilon^2)^{-\frac{1}{2}} = \mathrm{R}' \,.\, \mathrm{AC} \sin \mathrm{C} = \mathrm{R}' \,.\, \mathrm{AD};$$

also

$$\tan^2 \theta = \frac{1 \pm \epsilon \cos \Delta}{\epsilon \, (\epsilon \pm \cos \Delta)} = \frac{\mathrm{BF}}{\mathrm{AC} \,.\, \mathrm{CE}};$$

but BF = BF . AB = BC . BD, and AC . CE = CD . CB, so that

$$\tan^2 \theta = \mathrm{BD}/\mathrm{CD}.$$

The changes in the magnitude of θ and b are readily discussed by means of this construction. As Δ increases from 0 to $\frac{1}{2}\pi$, BC/CD increases from unity. It afterwards diminishes to unity, so that θ increases from 45° to Ω, and then diminishes to 45°. Also, as Δ increases from 0 to $\frac{1}{2}\pi$, AD increases from zero to $\epsilon_0 (1 + \epsilon_0^2)^{-\frac{1}{2}}$, and R′ also increases, so that b increases steadily in this range from 0 to $\mathrm{R}_0 (1 + \epsilon_0^2)^{-\frac{1}{2}}$. After this AD diminishes to zero. In the neighbourhood of the Principal Incidence we have $R \sin \Delta = R_0$,* so that in this region $b = R_0/BC$. As we go from the Principal Incidence, BC increases, so that b diminishes. Thus the maximum value of b is found *at* the Principal Incidence, and its value is $R_0 (1 + \epsilon_0^2)^{-\frac{1}{2}}$. The following table gives the position and magnitude of the maximum value of b for various incidences:—

ϕ.	10°.	30°.	60°.	65°.	67° 30′
θ (Realgar)………	45° 22′	48° 30′	66° 19′	73° 46′	83° 17′
b (Realgar)………	0 ·0011	0 ·0107	0 ·0521	0 ·0714	0 ·0786
θ (diamond) ……	45° 22′	48° 29′	65° 40′	75°	85° 33′
b (diamond) ……	0 ·0002	0 ·0023	0 ·0112	0 ·0143	0 ·0170

ϕ.	67° 40′.	68°.	70°.	75°.	80°.
θ (Realgar)………	82° 48′	80° 51′	73° 40′	64° 40′	56° 47′
b (Realgar)………	0 ·0785	0 ·0783	0 ·0769	0 ·0625	0 ·0539
θ (diamond) ……	88° 37′	84° 14′	75° 34′	64° 11′	56° 45′
b (diamond) ……	0 ·0172	0·0172	0 ·0161	0 ·0155	3 ·0119

The ridge of maxima is represented in plan and elevation in figs. 17 and 18 below.

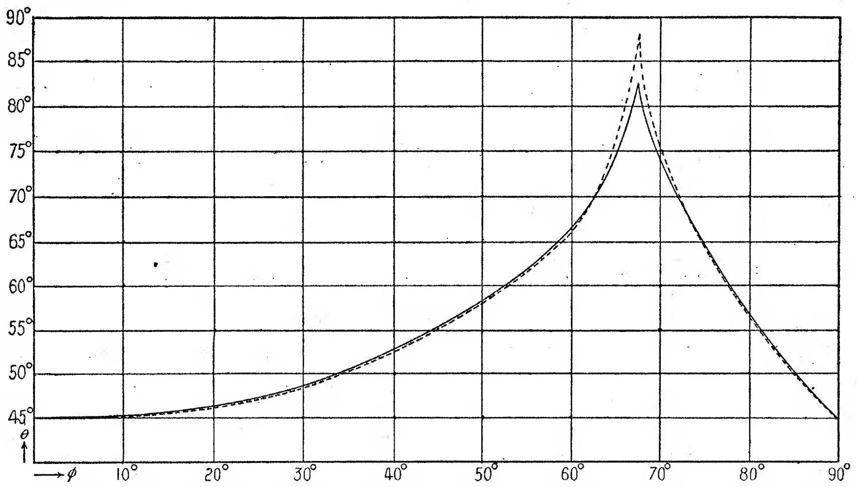

Fig. 17.—The continuous curve refers to realgar, the dotted curve to diamond.

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* See 'Roy. Soc. Proc.,' A, vol. 76, 1905, p. 61.

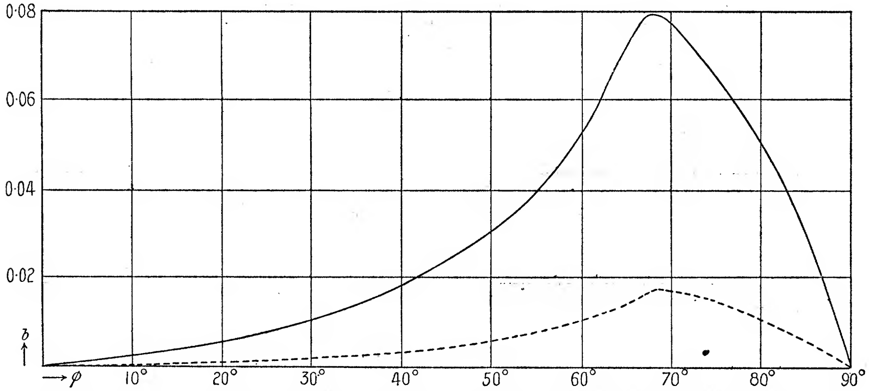

FIG. 18.—The continuous curve refers to realgar, the dotted curve to diamond.

In considering the variation of b with the incidence for a given azimuth, we take $b^2 = (2\epsilon)^{-1}\mathrm{R}^2 \sin\Delta \,.\, \sin 2\theta \,.\, \tan\gamma = \frac{1}{2}\mathrm{R}'\,\mathrm{R}\sin\Delta \,.\, \sin 2\theta \,.\, \tan\gamma$.

As the incidence increases from direct incidence to the Principal Incidence, $\mathrm{R}\sin\Delta$ increases throughout, except at the end where it is stationary.* Tan γ and R′ also increase, so that b increases from zero upwards. When the Principal Incidence is passed, γ decreases again to zero, so that b does likewise. It is evident then that b passes through a maximum between the Principal Incidence and grazing incidence, and that, as the azimuth increases to 90°, the position of this maximum draws closer and closer to the Principal Incidence. The following table gives the values of b for realgar and diamond for various incidences and azimuths :—

θ.	ϕ=10°.	30°.	60°.	65°.	67° 30′.	67° 40′.	68°.	70°.	75°.	80°.
10°	0·0004 0·0001	0·0032 0·0007	0·0102 0·0023	0·0134 0·0027	0·0138 0·0030	0·0139 0·0030	0·0139 0·0031	0·0143 0·0032	0·0123 0·0034	0·0082 0·0026
30°	0·0010 0·0001	0·0086 0·0019	0·0299 0·0056	0·0378 0·0076	0·0396 0·0085	0·0397 0·0085	0·0402 0·0086	0·0411 0·0087	0·0378 0·0095	0·0250 0·0082
45°	0·0011 0·0002	0·0106 0·0023	0·0417 0·0084	0·0517 0·0108	0·0560 0·0118	0·0562 0·0119	0·0569 0·0120	0·0585 0·0124	0·0527 0·0132	0·0430 0·0111
60°	0·0010 0·0001	0·0098 0·0021	0·0515 0·0105	0·0632 0·0131	0·0686 0·0149	0·0690 0·0150	0·0698 0·0150	0·0737 0·0150	0·0617 0·0153	0·0480 0·0119
75°	0·0006 0·0001	0·0059 0·0013	0·0475 0·0104	0·0665 0·0143	0·0765 0·0169	0·0770 0·0168	0·0778 0·0168	0·0766 0·0159	0·0564 0·0130	0·0334 0·0087
85°	0·0004 0·0001	0·0021 0·0009	0·0241 0·0053	0·0462 0·0117	0·0617 0·0170	0·0618 0·0170	0·0611 0·0170	0·0495 0·0136	0·0266 0·0066	0·0152 0·0034
89°	0 0	0·0004 0·0001	0·0052 0·0012	0·0100 0·0035	0·0123 0·0118	0·0124 0·0124	0·0123 0·0112	0·0107 0·0042	0·0056 0·0014	0·0039 0·0006

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* See 'Roy. Soc. Proc.,' A, vol. 76, 1905, p. 61.

These results are represented in figs. 19 and 20 below.

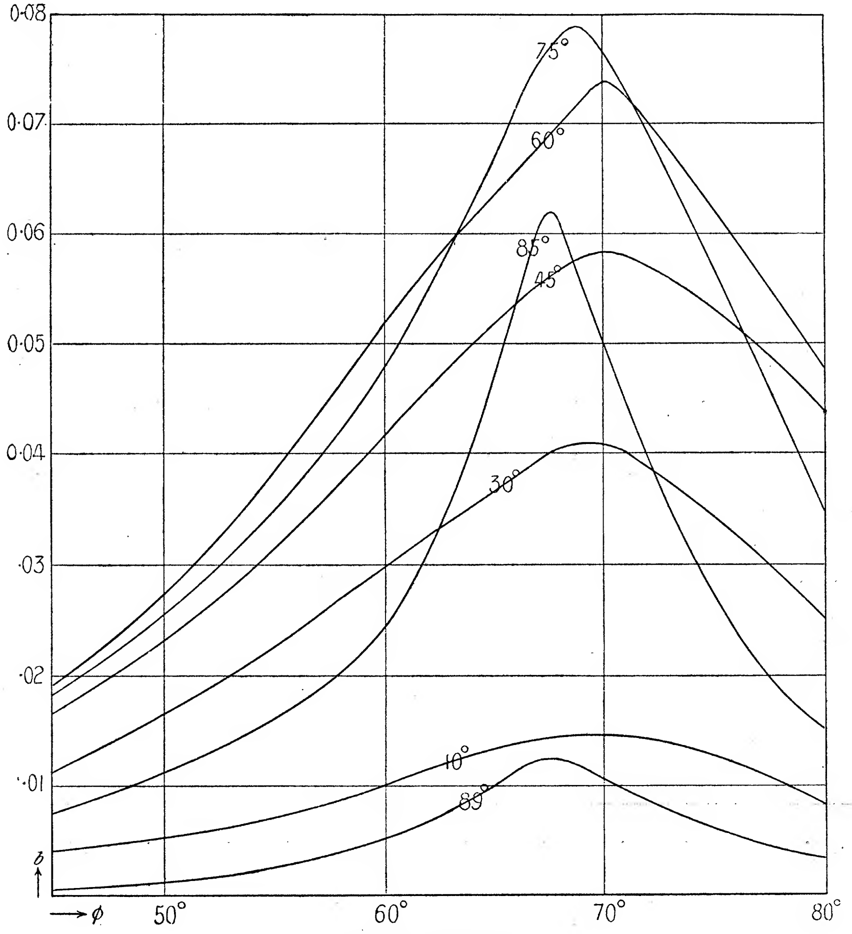

FIG. 19.—Realgar.

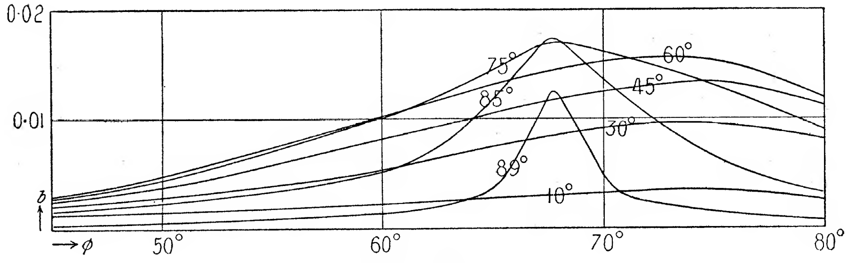

FIG. 20.—Diamond.

The area of the elliptic orbit is

$$\pi ab = (2\epsilon)^{-1}\pi R^2 \sin \Delta \sin 2\theta = \tfrac{1}{2}\pi R'R \sin \Delta \,.\, \sin 2\theta.$$

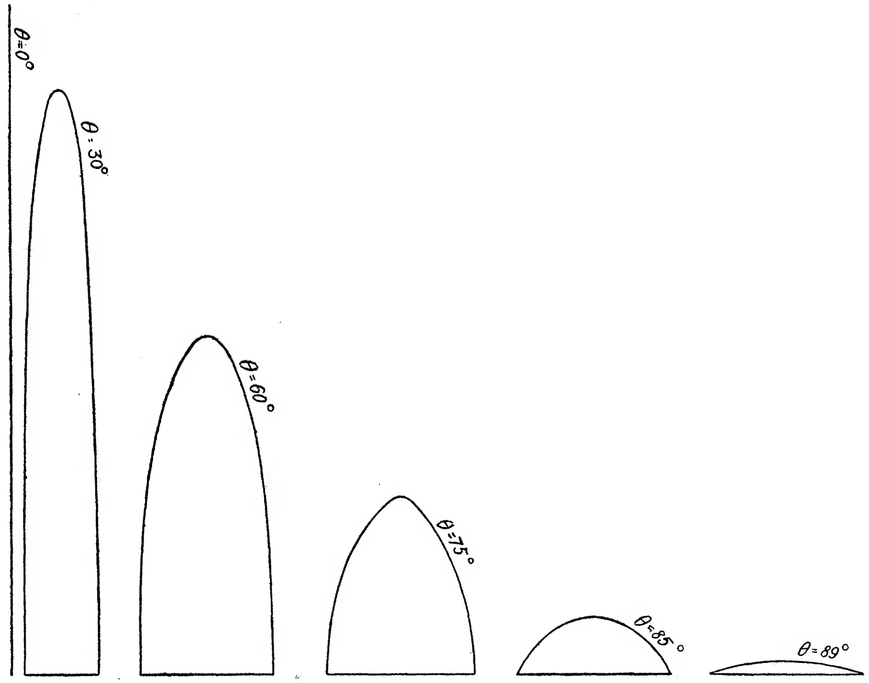

FIG. 21.—Realgar.

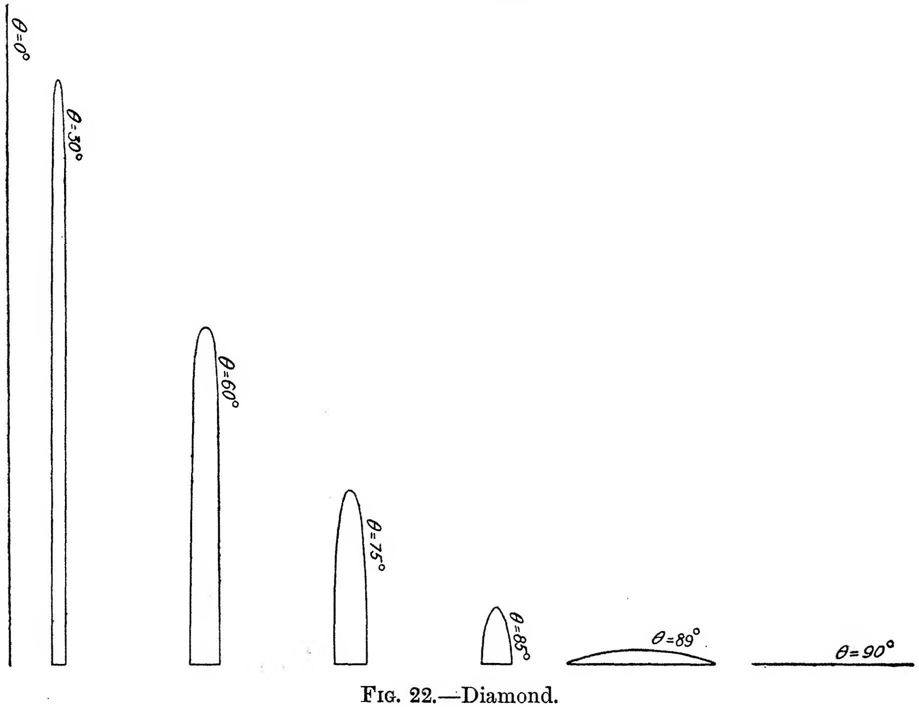

FIG. 22.—Diamond.

For a given incidence this is proportional to sin 2θ, it therefore grows from zero to a maximum when $\theta = 45°$, and diminishes again to zero as θ approaches 90°.

For a given azimuth the area is proportional to R′R sin Δ. As ϕ increases from zero to the Principal Incidence, R′ increases, and R sin Δ increases, except near the Principal Incidence, where it is stationary. Hence the area of the ellipse constantly increases within this range. In the neighbourhood of the Principal Incidence R sin Δ is constant, so that the area is proportional to R′, and continues to increase throughout a short range beyond the Principal Incidence. When this region is passed, sin Δ diminishes rapidly to zero, so that the area diminishes to zero, passing through a maximum between the Principal Incidence and grazing incidence.

Figs. 21 and 22 represent half of the elliptic orbit, drawn to scale for various azimuths *at the Principal Incidence.*